Digital Signal Processing

Homework Assignment Number 3 Solutions

Q1. From the table of z-transforms

$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-0.5z^{-1}}$$

Hence

$$\begin{aligned}
X(e^{j\omega}) &= \frac{1}{1-0.5e^{-j\omega}} \\
&= \frac{1-0.5e^{j\omega}}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} \\
&= \frac{1-0.5\cos(\omega)-0.5j\sin(\omega)}{1-0.5e^{-j\omega}-0.5e^{j\omega}+0.25} \\
&= \frac{1-0.5\cos(\omega)-0.5j\sin(\omega)}{1.25-\cos(\omega)} \\
&= \frac{\sqrt{(1-0.5\cos(\omega))^2+.25\sin(\omega)^2}}{1.25-\cos(\omega)} e^{j\tan^{-1}(\frac{-0.5\sin(\omega)}{1-0.5\cos(\omega)})} \\
&= \frac{\sqrt{1.25-\cos(\omega)}}{1.25-\cos(\omega)} e^{j\tan^{-1}(\frac{-0.5\sin(\omega)}{1-0.5\cos(\omega)})} \\
&= \frac{1}{\sqrt{1.25-\cos(\omega)}} e^{j\tan^{-1}(\frac{-0.5\sin(\omega)}{1-0.5\cos(\omega)})}
\end{aligned}$$

Q2. The averaging filter is defined as follows:

$$y(n) = \frac{1}{N}\sum_{k=n-N-1}^{n} x(k)$$

(a) Hence, if $x(n) = \delta(n)$,

$$h(n) = \begin{cases} \frac{1}{N} & \text{if } n \in [0, N-1]; \\ 0 & \text{otherwise} \end{cases}$$

(b) Using the table of z-transforms, we obtain

$$H(z) = \begin{cases} \frac{1-z^{-N}}{1-z^{-1}} & \text{if } z \neq 1; \\ N & \text{if } z = 1 \end{cases}$$

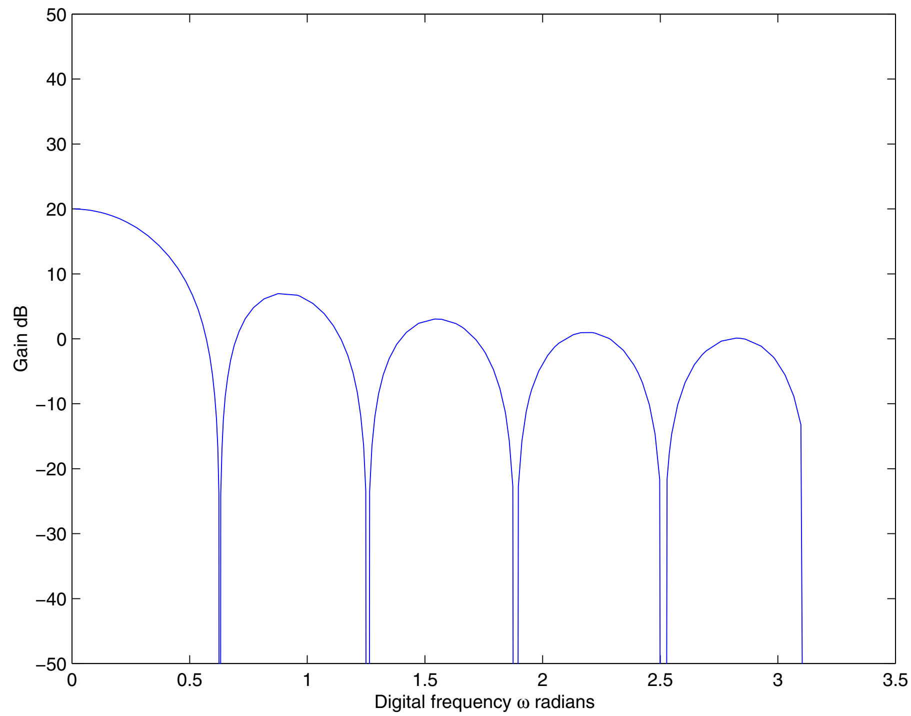
Note that this transfer function does not have a pole at $z = 1$ since $H(z)$ has a finite value of N at $z = 1$.

(c) The following matlab commands were used to obtain the magnitude plot for the case of $N = 10$:

```
>> N=10;z=tf('z',1);H=(1-z^(-N))/(1-z^(-1))

Transfer function:
 z^11 - z
-----------
z^11 - z^10

Sampling time: 1
>> [MAG,PHASE,W]=bode(H);
```

```
>> clf
>> plot(W,20*log10(squeeze(MAG)))
>> axis([0 3.5 -50 50])
>> xlabel('Digital frequency \omega radians')
>> ylabel('Gain dB')
```

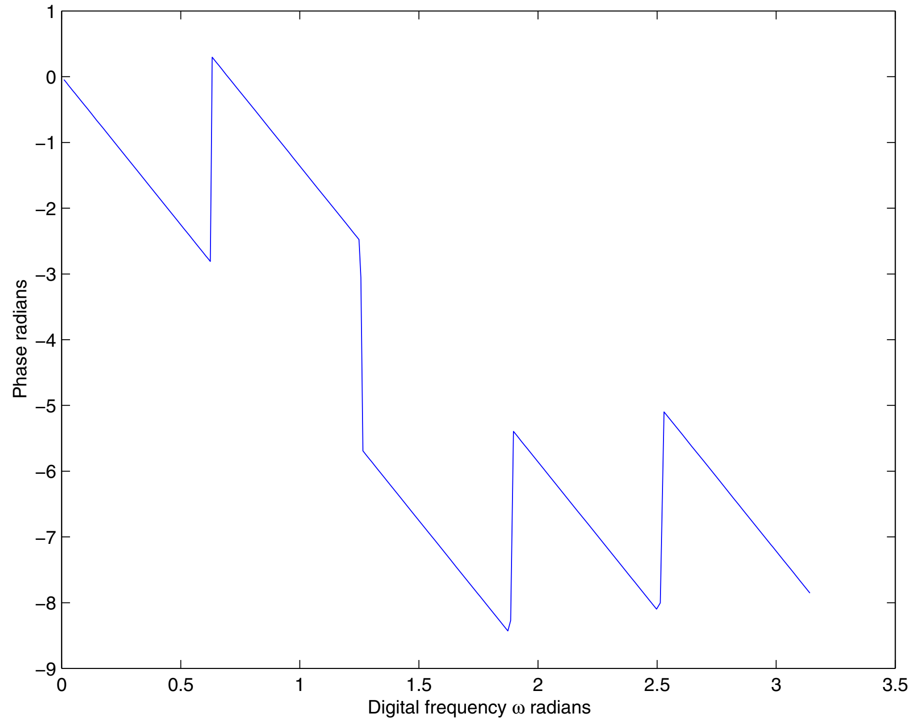
This gave the plot



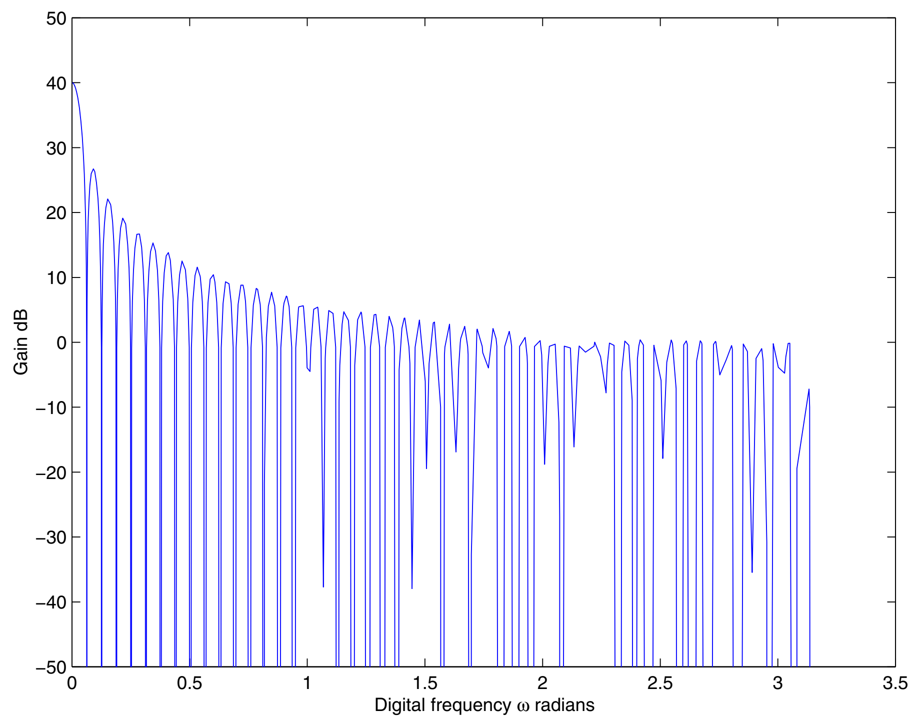
Also, the corresponding phase plot was generated with the commands

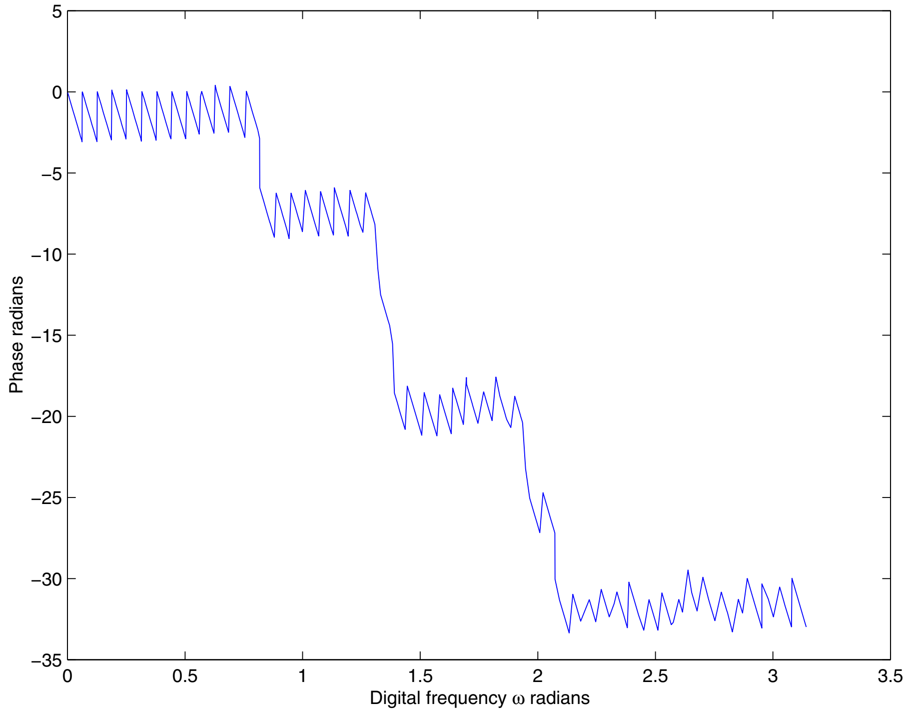
```
>> plot(W,squeeze(PHASE)*pi/180)
>> xlabel('Digital frequency \omega radians')
>> ylabel('Phase radians')
```

This gave the plot



The above process was repeated with $N = 100$. This gave the plots





From this plots, we can see that in the first case of $N = 10$, the filter acts as low pass filter with a quite wide pass band and a roll off which is not very sharp. In the second case of $N = 100$, the filter has a very narrow pass band and a much sharper rolloff.

Q3. (a) Taking z-transforms, we obtain

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = -\frac{1}{16}X(z) + \frac{7}{16}z^{-2}X(z)$$

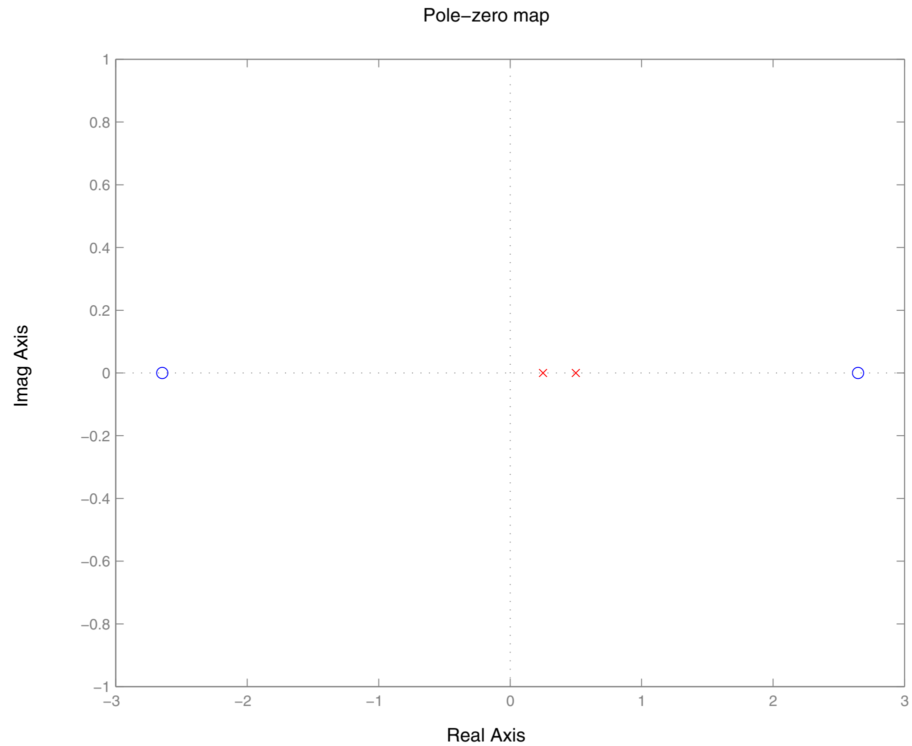
Hence, we obtain the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{16} + \frac{7}{16}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{-\frac{1}{16}z^2 + \frac{7}{16}}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

The pole-zero diagram is generated in matlab as follows:

```
z=tf('z',1);H=(-1/16*z^2+7/16)/(z^2-3/4*z+1/8);pzmap(H)
```

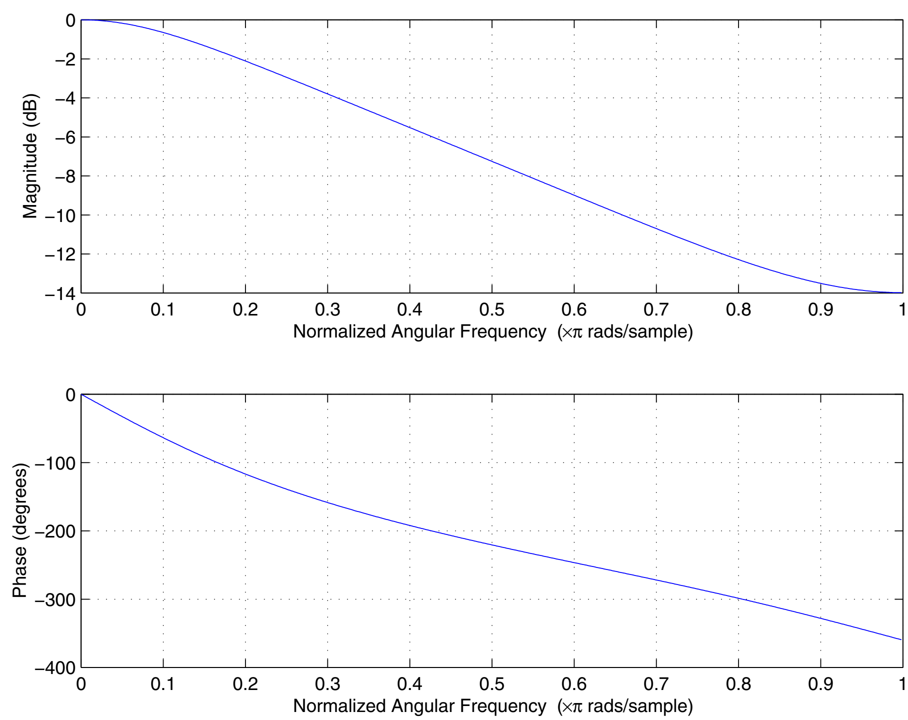
This gave the following plot



(b) The frequency response is plotted using matlab as follows

```
>> B=[-1/16 0 7/16];A=[1 -3/4 1/8];freqz(B,A)
```

This gave the following plot



From this plot, we can see that the filter is a low pass filter.

(c) To find the unit sample response, we need to find the inverse z-tranform of $H(z)$ using the partial fraction expansion method. Hence, we write

$$\frac{H(z)}{z} = \frac{-\frac{1}{16}z^2 + \frac{7}{16}}{z(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{a_{-1}^{\{1\}}}{z} + \frac{a_{-1}^{\{2\}}}{z-0.5} + \frac{a_{-1}^{\{3\}}}{z-0.25}$$

where

$$a_{-1}^{\{1\}} = \frac{\frac{7}{16}}{(-\frac{1}{2})(-\frac{1}{4})} = 3.5; \quad a_{-1}^{\{2\}} = \frac{-\frac{1}{16}.5^2 + \frac{7}{16}}{.5(.5-\frac{1}{4})} = 3.3750; \quad a_{-1}^{\{3\}} = \frac{-\frac{1}{16}.25^2 + \frac{7}{16}}{.25(.25-\frac{1}{2})} = -6.9375$$

Hence,

$$H(z) = 3.5 + \frac{3.3750z}{z-0.5} - \frac{6.9375z}{z-0.25}$$

and therefore using our table of standard z-transfroms, we obtain the unit sample response

$$h(n) = 3.5\delta(n) + 3.3750 \times 0.5^n - 6.9375 \times 0.25^n$$

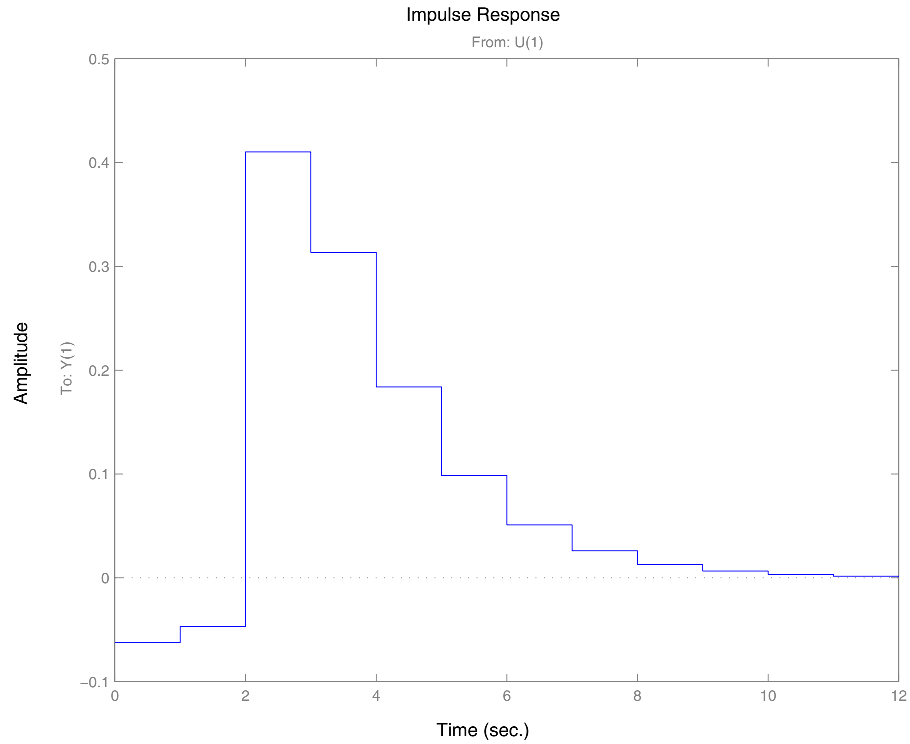
To verify our partial fraction expansion coefficients, we use the matlab commands:

```
>> B=[-1/16 0 7/16];A=[1 -3/4 1/8];[R,P,K]=residuez(B,A)

R =

    3.3750
   -6.9375


P =

    0.5000
    0.2500


K =

    3.5000
```

This verifies that our calculations were correct. Also, we can directly plot the impulse response using matlab as follows:

```
>> z=tf('z',1);H=(-1/16*z^2+7/16)/(z^2-3/4*z+1/8);impulse(H)
```

This gave the following plot



which agrees with our analytic calculation.

Q4. To find the DTFT, we first find the z-transform of $x(n)$ from the table of z-transforms:

$$X(z) = \begin{cases} \frac{1-z^{-M}}{1-z^{-1}} & \text{if } z \neq 1; \\ M & \text{if } z = 1 \end{cases}$$

Hence, the DTFT is

$$X(e^{j\omega}) = \begin{cases} \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} & \text{if } \omega \neq 0; \\ M & \text{if } \omega = 0 \end{cases}$$

To obtain the corresponding DFT, we sample the DTFT at multiples of the fundamental frequency $\omega_0 = 2\pi/N$; i.e.,

$$DFT_N\{x(n)\}(k) = X(e^{j\omega})|_{\omega=2\pi k/N} = \begin{cases} \frac{1-e^{-j2\pi kM/N}}{1-e^{-j2\pi k/N}} & \text{if } k \neq 0; \\ M & \text{if } k = 0 \end{cases}$$

$k = 0, 1, \ldots, N-1$.

Note that the zeros of the DTFT occur at digital frequencies $\omega \neq 0$ such that

$$1 = e^{-j\omega M}$$

That is,

$$-2\pi k = -\omega M; \quad k = 1, 2, \ldots, M-1$$

Hence, the zeros occur at the digital frequencies

$$\omega = 2\pi k/M; \quad k = 1, 2, \ldots, M-1$$